

2-KILLING VECTOR FIELDS ON STANDARD STATIC SPACE-TIMES

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ABSTRACT. This article provides a short study of 2-Killing vector fields on warped product manifolds as well as on standard static spacetimes. Conditions on a 2-Killing vector field on a warped product manifold to be parallel are derived. The article also provides some results on the curvature of warped product manifolds in terms of 2-Killing vector fields. Finally, we apply some of the results on standard static spacetimes and the warped product manifold $I_f \times \mathbb{R}$ as an example of those manifolds.

1. AN INTRODUCTION

Killing vector fields have a well-known geometrical and physical interpretations and have been studied on Riemannian and pseudo-Riemannian manifolds for a long time. The number of independent Killing vector fields measures the degree of symmetry of a Riemannian manifold. Thus the problems of existence and characterization of Killing vector fields are important and are widely discussed in both geometry and physics [1–4, 9–11, 13, 16, 17]

Generalization of Killing vector fields has a long history in mathematics for different scale and purpose [6–8, 12]. In [15], the concept of 2-Killing vector fields as a new generalization of Killing vector fields is defined and studied on Riemannian manifolds. The relations between 2-Killing vector fields, curvature and monotone vector fields are obtained. Finally, a characterization of 2-Killing vector field on \mathbb{R}^n is derived.

As far as we know, there is no study of this concept neither in warped product manifolds nor in spacetimes. This article provides a short study of 2-killing vector fields on such spaces. The next section presents the general details of warped product manifolds and Killing vector fields that should be known. Most of results in this section are well known and therefore some of the proofs are omitted. Section 3 is the core of this article. In this section the relation between 2-Killing vector fields on a Riemannian warped product manifold and 2-Killing vector fields on the product factors is discussed in the following results. The following theorem represents an important and helpful identity.

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Theorem L. *et $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$. Then*

$$\begin{aligned} (\mathcal{L}_\zeta \mathcal{L}_\zeta g)(X, Y) &= \left(\mathcal{L}_{\zeta_1}^1 \mathcal{L}_{\zeta_1}^1 g_1 \right) (X_1, Y_1) + f^2 \left(\mathcal{L}_{\zeta_2}^2 \mathcal{L}_{\zeta_2}^2 g_2 \right) (X_2, Y_2) \\ &\quad + 4f\zeta_1(f) \left(\mathcal{L}_{\zeta_2}^2 g_2 \right) (X_2, Y_2) + 2f\zeta_1(\zeta_1(f)) g_2(X_2, Y_2) \\ &\quad + 2\zeta_1(f) \zeta_1(f) g_2(X_2, Y_2) \end{aligned}$$

for any vector fields $X, Y \in \mathfrak{X}(M_1 \times_f M_2)$.

The proof of this result contains a large computations that have been done using previous results on warped product manifolds (see appendix A). As an immediate consequence the relation between 2-Killing vector fields on warped product manifolds and those on product factors is given.

Conditions on 2-Killing vector fields to be parallel vector fields is considered in the following theorem.

Theorem L. *et $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$. Then*

- (1) $\zeta = \zeta_1 + \zeta_2$ is parallel if ζ_i is a 2-Killing vector field, $\text{Ric}^i(\zeta_i, \zeta_i) \leq 0$, $i = 1, 2$ and f is constant.
- (2) $\zeta = \zeta_1$ is parallel if ζ_1 is a 2-Killing vector field, $\text{Ric}^1(\zeta_1, \zeta_1) \leq 0$, and $X_1(f) = 0$.
- (3) $\zeta = \zeta_2$ is parallel if ζ_2 is a 2-Killing vector field, $\text{Ric}^2(\zeta_2, \zeta_2) \leq 0$, and f is constant.

The following theorem also provides some results on the curvature of warped product manifolds in terms of 2-Killing vector fields.

Theorem L. *et $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a non-trivial 2-Killing vector field. If $D_\zeta \zeta$ is parallel along a curve γ , then*

$$K(\zeta, \dot{\gamma}) \geq 0.$$

Finally, in section 4, we apply these results on standard static spacetimes and give an example. For instance, the following result is obtained.

Theorem L. *et $\bar{M} = I_f \times M$ be a standard static space-time with the metric $\bar{g} = -f^2 dt^2 \oplus g$. Suppose that $u : I \rightarrow \mathbb{R}$ is smooth and ζ is a vector field on F . Then $\bar{\zeta} = u\partial_t + \zeta$ is a 2-Killing vector field on \bar{M} if one of the the following conditions is satisfied:*

- (1) ζ is Killing on M , $u = a$ and $f\zeta(f) = b$ where $a, b \in \mathbb{R}$.
- (2) ζ is Killing on M , $u = (rt + s)^{\frac{1}{3}}$ and $\zeta(f) = 0$ where $r, s \in \mathbb{R}$.

Also, the converse of this result is discussed.

2. PRELIMINARIES

Let (M_i, g_i, D_i) , $i = 1, 2$ be two C^∞ Riemannian manifolds equipped with Riemannian metrics g_i where D_i is the Levi-Civita connection of the metric g_i . Let $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$ be the natural projection maps of the Cartesian product $M_1 \times M_2$ onto M_1 and M_2 respectively. Also, let $f : M_1 \rightarrow (0, \infty)$ be a positive real-valued smooth function. The warped product manifold $M_1 \times_f M_2$

is the the product manifold $M_1 \times M_2$ equipped with the metric tensor $g = g_1 \oplus g_2$ defined by

$$g = \pi_1^* (g_1) \oplus (f \circ \pi_1)^2 \pi_2^* (g_2)$$

where $*$ denotes the pull-back operator on tensors [5, 14]. The function f is called the warping function of the warped product manifold $M_1 \times_f M_2$. In particular, if $f = 1$, then $M_1 \times_1 M_2 = M_1 \times M_2$ is the usual Cartesian product manifold. It is clear that the submanifold $M_1 \times \{q\}$ is isometric to M_1 for every $q \in M_2$. Also, $\{p\} \times M_2$ is homothetic to M_2 . Throughout this article we use the same notation for a vector field and for its lift to the product manifold.

Let D be the Levi-Civita connection of the metric tensor g . The following proposition is well-known [5].

Proposition 1. *Let $(M_1 \times_f M_2, g)$ be a Riemannian warped product manifold with warping function $f > 0$ on M_1 . Then*

- (1) $D_{X_1} Y = D_{X_1}^1 Y_1 \in \mathfrak{X}(M_1)$
- (2) $D_{X_1} Y_2 = D_{Y_2} X_1 = \frac{X_1(f)}{f} Y_2$
- (3) $D_{X_2} Y_2 = -f g_2(X_2, Y_2) \nabla f + D_{X_2}^2 Y_2$

for all $X_i, Y_i \in \mathfrak{X}(M_i)$, $i = 1; 2$ where ∇f is the gradient of f .

A vector field $\zeta \in \mathfrak{X}(M)$ on a manifold (M, g) with metric g is called a Killing vector field if

$$\mathcal{L}_\zeta g = 0$$

where \mathcal{L}_ζ is the Lie derivative on M with respect to ζ . One can redefine Killing vector fields using the following identity. Let ζ be a vector field, then

$$(2.1) \quad (\mathcal{L}_\zeta g)(X, Y) = g(D_X \zeta, Y) + g(X, D_Y \zeta)$$

for any vector fields $X, Y \in \mathfrak{X}(M)$. A good and simple characterization of Killing vector fields is given in the following proposition. The proof is direct using the symmetry in the above identity.

Lemma 1. *If (M, g, D) is a Riemannian manifold with Riemannian connection D . A vector field $\zeta \in \mathfrak{X}(M)$ is a Killing vector field if and only if*

$$(2.2) \quad g(D_X \zeta, X) = 0$$

for any vector field $X \in \mathfrak{X}(M)$.

Now we consider Killing vector fields on Riemannian warped product manifolds. The following simple result will help us to present a characterization of Killing vector fields on warped product manifolds.

Lemma 2. *Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the Riemannian warped product manifold $M_1 \times_f M_2$ with warping function f . Then for any vector field $X \in \mathfrak{X}(M_1 \times_f M_2)$ we have*

$$(2.3) \quad g(D_X \zeta, X) = g_1(D_{X_1}^1 \zeta_1, X_1) + f^2 g_2(D_{X_2}^2 \zeta_2, X_2) + f \zeta_1(f) \|X_2\|^2$$

Proof. Using Proposition 1, we get

$$\begin{aligned}
g(D_X \zeta, X) &= g_1(D_{X_1}^1 \zeta_1 - f g_2(X_2, \zeta_2) \nabla f, X_1) + f^2 g_2(D_{X_2}^2 \zeta_2 + \zeta_1(\ln f) X_2 \\
&\quad + X_1(\ln f) \zeta_2, X_2) \\
&= g_1(D_{X_1}^1 \zeta_1, X_1) - f g_2(X_2, \zeta_2) X_1(f) + f^2 g_2(D_{X_2}^2 \zeta_2, X_2) \\
&\quad + f \zeta_1(f) g_2(X_2, X_2) + f X_1(f) g_2(\zeta_2, X_2) \\
&= g_1(D_{X_1}^1 \zeta_1, X_1) + f^2 g_2(D_{X_2}^2 \zeta_2, X_2) + f \zeta_1(f) \|X_2\|^2
\end{aligned}$$

□

These two results give us a characterization of Killing vector fields on warped product manifolds. They are immediate consequence of the previous result.

Theorem 1. *Let $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the Riemannian warped product manifold $M_1 \times_f M_2$ with warping function f . Then ζ is a Killing vector field if one of the following conditions hold*

- (1) $\zeta = (\zeta_1, 0)$ and ζ_1 is a killing vector field on M_1 .
- (2) $\zeta = (0, \zeta_2)$ and ζ_2 is a killing vector field on M_2 .
- (3) ζ_i is a Killing vector field on $M_i, i = 1, 2$ and $\zeta_1(f) = 0$.

The converse of the above result is considered in following theorem.

Theorem 2. *Let $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a killing vector field on the warped product manifold $M_1 \times_f M_2$ with warping function f . Then*

- (1) ζ_1 is a Killing vector field on M_1 .
- (2) ζ_2 is a Killing vector field on M_2 if $\zeta_1(f) = 0$.

In [unal2012] the authors proved similar results in standard static spacetimes using the following proposition.

Proposition 2. *Let $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$ with warping function f . Then*

$$(2.4) \quad (\mathcal{L}_\zeta g)(X, Y) = \left(\mathcal{L}_{\zeta_1}^1 g_1 \right)(X_1, Y_1) + f^2 \left(\mathcal{L}_{\zeta_2}^2 g_2 \right)(X_2, Y_2) + 2f \zeta_1(f) g_2(X_2, Y_2)$$

where $\mathcal{L}_{\zeta_i}^i$ is the Lie derivative on M_i with respect to $\zeta_i, i = 1, 2$.

3. 2-KILLING VECTOR FIELDS

A vector field $\zeta \in \mathfrak{X}(M)$ is called a 2-Killing vector field on a Riemannian manifold (M, g) if

$$(3.1) \quad \mathcal{L}_\zeta \mathcal{L}_\zeta g = 0$$

where \mathcal{L} is the Lie derivative on M [15].

The following two results [15] are needed to exploit the above definition.

Proposition 3. *Let $\zeta \in \mathfrak{X}(M)$ be a vector field on a Riemannian manifold M . Then*

$$(3.2) \quad (\mathcal{L}_\zeta \mathcal{L}_\zeta g)(X, Y) = g(D_\zeta D_X \zeta - D_{[\zeta, X]} \zeta, Y) + g(X, D_\zeta D_Y \zeta - D_{[\zeta, Y]} \zeta) + 2g(D_X \zeta, D_Y \zeta)$$

for any vector fields $X, Y \in \mathfrak{X}(M)$.

The following result is quite direct and helpful.

Corollary 1. *A vector field ζ is 2-Killing if and only if*

$$(3.3) \quad R(\zeta, X, \zeta, X) = g(D_X \zeta, D_X \zeta) + g(D_X D_\zeta \zeta, X)$$

for any vector field $X \in \mathfrak{X}(M)$.

The symmetry of equation (3.2), shows that ζ is 2-Killing if and only if

$$g(D_\zeta D_X \zeta - D_{[\zeta, X]} \zeta, X) + g(D_X \zeta, D_X \zeta) = 0$$

Example 1. Let M be the Euclidean plane (\mathbb{R}^2, ds^2) where $ds^2 = dx^2 + dy^2$. A vector field $\zeta = u\partial_x + v\partial_y \in \mathfrak{X}(M)$ is 2-Killing if

$$0 = (\mathcal{L}_\zeta^I \mathcal{L}_\zeta^I g_I)(X, Y)$$

for any vector fields X, Y , where \mathcal{L}_ζ is the Lie derivative on \mathbb{R}^2 with respect to ζ . Now it is easy to show that ζ is 2-Killing vector field on M if and only if

$$\begin{aligned} uu_{xx} + 2u_x^2 &= 0 \\ vv_{yy} + 2v_y^2 &= 0 \end{aligned}$$

By making use of the above proposition one can get sufficient and necessarily conditions for a vector field $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ to be 2-killing on the Riemannian warped product manifold $M_1 \times_f M_2$. The following theorem represents a similar such .

Theorem 3. Let $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$. Then

$$\begin{aligned} (\mathcal{L}_\zeta \mathcal{L}_\zeta g)(X, Y) &= \left(\mathcal{L}_{\zeta_1}^1 \mathcal{L}_{\zeta_1}^1 g_1 \right)(X_1, Y_1) + f^2 \left(\mathcal{L}_{\zeta_2}^2 \mathcal{L}_{\zeta_2}^2 g_2 \right)(X_2, Y_2) \\ &\quad + 4f\zeta_1(f) \left(\mathcal{L}_{\zeta_2}^2 g_2 \right)(X_2, Y_2) + 2\zeta_1(f) \zeta_1(f) g_2(X_2, Y_2) \\ &\quad + 2\zeta_1(f) \zeta_1(f) g_2(X_2, Y_2) \end{aligned}$$

for any vector fields $X, Y \in \mathfrak{X}(M_1 \times_f M_2)$.

Proof. See Appendix A □

Corollary 2. Let $\zeta = (\zeta_1, \zeta_2) \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$. Then ζ is a 2-killing vector field on $M_1 \times_f M_2$ if one of the following conditions holds

- (1) the vector field ζ_i is a 2-Killing vector field on $M_i, i = 1, 2$, and $\zeta_1(f) = 0$
- (2) $\zeta = (0, \zeta_2)$ and ζ_2 is a 2-Killing vector field on M_2 .

Theorem 4. Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$. Then

- (1) $\zeta = \zeta_1 + \zeta_2$ is parallel if ζ_i is a 2-Killing vector field, $\text{Ric}^i(\zeta_i, \zeta_i) \leq 0$, $i = 1, 2$ and f is constant.
- (2) $\zeta = \zeta_1$ is parallel if ζ_1 is a 2-Killing vector field, $\text{Ric}^1(\zeta_1, \zeta_1) \leq 0$, and $\zeta_1(f) = 0$.
- (3) $\zeta = \zeta_2$ is parallel if ζ_2 is a 2-Killing vector field, $\text{Ric}^2(\zeta_2, \zeta_2) \leq 0$, and f is constant.

Proof. Suppose that

$$\{e_1, e_2, \dots, e_m\}$$

is an orthonormal frame in $T_p M_1$ and

$$\{e_{m+1}, e_{m+2}, \dots, e_{m+n}\}$$

is an orthonormal frame in $T_q M_2$ for some point $(p, q) \in M_1 \times M_2$. Then

$$\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{m+n}\}$$

is an orthonormal frame in $T_{(p,q)}(M_1 \times M_2)$ where

$$\bar{e}_i = \begin{cases} e_i & 1 \leq i \leq m \\ \frac{1}{f} e_i & m+1 \leq i \leq m+n \end{cases}$$

Thus for any vector field $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ we have

$$\begin{aligned} \text{Tr}(g(D\zeta, D\zeta)) &= \sum_{i=1}^{m+n} g(D\bar{e}_i \zeta, D\bar{e}_i \zeta) \\ (3.4) \quad &= \sum_{i=1}^m g(D_{e_i} \zeta, D_{e_i} \zeta) + \frac{1}{f^2} \sum_{i=m+1}^{m+n} g(D_{e_i} \zeta, D_{e_i} \zeta) \end{aligned}$$

Using Proposition 1, the first term is given by

$$\begin{aligned} \sum_{i=1}^m g(D_{e_i} \zeta, D_{e_i} \zeta) &= \sum_{i=1}^m g(D_{e_i}^1 \zeta_1 + e_i(\ln f) \zeta_2, D_{e_i}^1 \zeta_1 + e_i(\ln f) \zeta_2) \\ &= \sum_{i=1}^m g(D_{e_i}^1 \zeta_1, D_{e_i}^1 \zeta_1) + \sum_{i=1}^m g(e_i(\ln f) \zeta_2, e_i(\ln f) \zeta_2) \\ &= \text{Tr}(g_1(D^1 \zeta_1, D^1 \zeta_1)) + \|\zeta_2\|^2 \sum_{i=1}^m (e_i(\ln f))^2 \\ (3.5) \quad &= \text{Tr}(g_1(D^1 \zeta_1, D^1 \zeta_1)) + \|\zeta_2\|^2 \|\nabla f\|^2 \end{aligned}$$

and the second term is given by

$$\begin{aligned} &\frac{1}{f^2} \sum_{i=m+1}^{m+n} g(D_{e_i} \zeta, D_{e_i} \zeta) \\ &= \frac{1}{f^2} \sum_{i=m+1}^{m+n} g(\zeta_1(\ln f) e_i + D_{e_i}^2 \zeta_2 - f g_2(e_i, \zeta_2) \nabla f, \zeta_1(\ln f) e_i \\ (3.6) \quad &+ D_{e_i}^2 \zeta_2 - f g_2(e_i, \zeta_2) \nabla f) \\ &= n(\zeta_1(\ln f))^2 + \sum_{i=m+1}^{m+n} g_2(D_{e_i}^2 \zeta_2, D_{e_i}^2 \zeta_2) + \|\nabla f\|^2 \sum_{i=m+1}^{m+n} (g_2(e_i, \zeta_2))^2 \end{aligned}$$

$$\begin{aligned} &\frac{1}{f^2} \sum_{i=m+1}^{m+n} g(D_{e_i} \zeta, D_{e_i} \zeta) \\ (3.7) \quad &= \frac{n}{f^2} (\zeta_1(f))^2 + \text{Tr}(g_2(D^2 \zeta_2, D^2 \zeta_2)) + \|\nabla f\|^2 \|\zeta_2\|^2 \end{aligned}$$

By using equations (3.5) and (3.7), equation (3.4) becomes

$$\begin{aligned}
 & Tr(g(D\zeta, D\zeta)) \\
 (3.8) \quad & = Tr(g_1(D^1\zeta_1, D^1\zeta_1)) + Tr(g_2(D^2\zeta_2, D^2\zeta_2)) + 2\|\zeta_2\|^2 \|\nabla f\|^2 \\
 (3.9) \quad & + \frac{n}{f^2} (\zeta_1(f))^2
 \end{aligned}$$

Now suppose that ζ_i is a 2-Killing vector field and $Ric^i(\zeta_i, \zeta_i) \leq 0$, then ζ_i is a parallel vector field with respect to the metric g_i and so

$$Tr(g_1(D^1\zeta_1, D^1\zeta_1)) = Tr(g_2(D^2\zeta_2, D^2\zeta_2)) = 0$$

Then for a constant function f we have

$$Tr(g(D\zeta, D\zeta)) = 0$$

Thus ζ is a parallel vector field with respect to the metric g . One easily can prove the last two assertions using equation 3.9. \square

Corollary 3. *Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a vector field on the warped product manifold $M_1 \times_f M_2$. Then*

$$\begin{aligned}
 & Tr(g(D\zeta, D\zeta)) \\
 & = Tr(g_1(D^1\zeta_1, D^1\zeta_1)) + Tr(g_2(D^2\zeta_2, D^2\zeta_2)) + 2\|\zeta_2\|^2 \|\nabla f\|^2 \\
 & + \frac{n}{f^2} \zeta_1(f) \zeta_1(f)
 \end{aligned}$$

Theorem 5. *Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a non-trivial 2-Killing vector field. If $D_\zeta \zeta$ is parallel along a curve γ , then*

$$K(\zeta, \dot{\gamma}) \geq 0.$$

Proof. Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a non-trivial 2-Killing vector field, then

$$\begin{aligned}
 0 & = g(D_\zeta D_X \zeta, Y) - g(D_{[\zeta, X]} \zeta, Y) + 2g(D_X \zeta, D_Y \zeta) \\
 & + g(X, D_\zeta D_Y \zeta) - g(X, D_{[\zeta, Y]} \zeta)
 \end{aligned}$$

for any vector fields $X, Y \in \mathfrak{X}(M_1 \times_f M_2)$. Let $X = Y = T = \dot{\gamma}$, then

$$\begin{aligned}
 g(D_\zeta D_T \zeta, T) - g(D_{[\zeta, T]} \zeta, T) + g(D_T \zeta, D_T \zeta) & = 0 \\
 g(D_\zeta D_T \zeta - D_{[\zeta, T]} \zeta, T) & = -g(D_T \zeta, D_T \zeta)
 \end{aligned}$$

Since $D_\zeta \zeta$ is parallel along a curve γ , $D_T D_\zeta \zeta = 0$ and hence

$$\begin{aligned}
 g(R(\zeta, T) \zeta, T) & = -g(D_T \zeta, D_T \zeta) \\
 R(\zeta, T, T, \zeta) & = -g(D_T \zeta, D_T \zeta) \\
 K(\zeta, \dot{\gamma}) & = \|D_T \zeta\|^2 * A(\zeta, \dot{\gamma}) \geq 0
 \end{aligned}$$

where $A(\zeta, \dot{\gamma})$ is area of the parallelogram generated by ζ and $\dot{\gamma}$. \square

The above result may be proved using Corollary 1 as follows. Let $\zeta \in \mathfrak{X}(M_1 \times_f M_2)$ be a non-trivial 2-Killing vector field, then

$$\begin{aligned}
 R(\zeta, T, \zeta, T) & = g(D_T \zeta, D_T \zeta) + g(D_T D_\zeta \zeta, T) \\
 & = \|D_T \zeta\|^2 + 0 \\
 & = \|D_T \zeta\|^2 \geq 0
 \end{aligned}$$

Moreover, if $D_\zeta \zeta = 0$, then $K(\zeta, X) \geq 0$ for any vector field $X \in \mathfrak{X}(M_1 \times_f M_2)$.

4. 2-KILLING VECTOR FIELDS ON STANDARD STATIC SPACETIMES

Let (M, g) be an n -dimensional Riemannian manifold and $f : M \rightarrow (0, \infty)$ be a smooth function. Then $(n+1)$ -dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -f^2 dt^2 \oplus g$$

is called a standard static space-time and is denoted by $\bar{M} = I_f \times M$, where dt^2 is the Euclidean metric tensor on I . The Einstein static universe is a good example of standard static space-times [4, 9].

Theorem 6. *Let $\bar{M} = I_f \times M$ be a standard static space-time with the metric $\bar{g} = -f^2 dt^2 \oplus g$. Suppose that $u : I \rightarrow \mathbb{R}$ is smooth on I . Then $\bar{\zeta} = u\partial_t + \zeta$, $\zeta \in \mathfrak{X}(M)$ is a 2-Killing vector field on \bar{M} if one of the the following conditions is satisfied:*

- (1) ζ is Killing on M , $u = a$ and $f\zeta(f) = b$ where $a, b \in \mathbb{R}$.
- (2) ζ is Killing on M , $u = (rt + s)^{\frac{1}{3}}$ and $\zeta(f) = 0$ where $r, s \in \mathbb{R}$.

Proof. Let $\bar{X} = x\partial_t + X \in \mathfrak{X}(\bar{M})$ and $\bar{Y} = y\partial_t + Y \in \mathfrak{X}(\bar{M})$ be any vector fields on \bar{M} where $X, Y \in \mathfrak{X}(M)$ and x, y are smooth real-valued functions on I . Using Theorem 3, we have

$$\begin{aligned} & (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\bar{X}, \bar{Y}) \\ &= (\mathcal{L}_{\zeta} \mathcal{L}_{\zeta} g)(X, Y) + f^2 (\mathcal{L}_{u\partial_t}^I \mathcal{L}_{u\partial_t}^I g_I)(x\partial_t, y\partial_t) + 4f\zeta(f) (\mathcal{L}_{\zeta_2}^2 g_2)(x\partial_t, y\partial_t) \\ & \quad + 2f\zeta(\zeta(f)) g_I(x\partial_t, y\partial_t) + 2\zeta(f)\zeta(f) g_I(x\partial_t, y\partial_t) \end{aligned}$$

Note that for a vector $u\partial_t$ field on I , we have

$$\begin{aligned} \mathcal{L}_{\zeta} g_I(x\partial_t, y\partial_t) &= 2\dot{u} g_I(x\partial_t, y\partial_t) \\ \mathcal{L}_{\zeta} \mathcal{L}_{\zeta} g_I(x\partial_t, y\partial_t) &= (2u\ddot{u} + 4\dot{u}^2) g_I(x\partial_t, y\partial_t) \end{aligned}$$

Then

$$\begin{aligned} & (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\bar{X}, \bar{Y}) \\ &= (\mathcal{L}_{\zeta} \mathcal{L}_{\zeta} g)(X, Y) + f^2 (2u\ddot{u} + 4\dot{u}^2) g_I(x\partial_t, y\partial_t) + 8\dot{u} f\zeta(f) g_I(x\partial_t, y\partial_t) \\ (4.1) \quad & + 2\zeta(f\zeta(f)) g_I(x\partial_t, y\partial_t) \end{aligned}$$

The vector field ζ is 2-Killing on M and the function u in both conditions 1 and 2 is a solution of

$$(2u\ddot{u} + 4\dot{u}^2) = 0$$

Thus equation (4.1) becomes

$$(4.2) \quad (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\bar{X}, \bar{Y}) = 2[4f\zeta(f)\dot{u} + \zeta(f\zeta(f))] g_I(x\partial_t, y\partial_t)$$

Finally, condition 1 implies that $\dot{u} = \zeta(f\zeta(f)) = 0$ and condition 2 implies that $\zeta(f) = 0$. Consequently, condition 1 or condition 2 implies that

$$(\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\bar{X}, \bar{Y}) = 0$$

and so $\bar{\zeta}$ is 2-Killing on \bar{M} . □

The converse of the above theorem is considered in the following corollary. The proof is direct.

Corollary 4. *Let \bar{M} be as above and $\bar{\zeta} = u\partial_t + \zeta$ be a 2-Killing vector field on \bar{M} . Then ζ is a 2-killing vector field on M . Moreover, the vector field $u\partial_t$ is a 2-Killing vector field on I if $\zeta(f) = 0$.*

Example 2. *Let $\zeta = u(t)\partial_t + v(x)\partial_x$ be a vector field on the warped product manifold $\bar{M} = I_f \times \mathbb{R}$ with warping function f and with metric $ds^2 = f^2 dt^2 + dx^2$. To prove that ζ is a 2-Killing vector field we can use equation (4.1). Let $\bar{X} = x\partial_t + X$ and $\bar{Y} = y\partial_t + Y$ be two vector fields on \bar{M} , then*

$$(4.3) \quad (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\bar{X}, \bar{Y}) = (\mathcal{L}_{\zeta} \mathcal{L}_{\zeta} g)(X, Y) + f^2(2u\ddot{u} + 4\dot{u}^2)g_I(x\partial_t, y\partial_t) \\ + 8\dot{u}f\zeta(f)g_I(x\partial_t, y\partial_t) + 2\zeta(f\zeta(f))g_I(x\partial_t, y\partial_t),$$

where $\zeta = v(x)\partial_x$ and $g = dx^2$. Now it is easy to show that

$$\begin{aligned} \zeta(f) &= vf', \quad \zeta(f\zeta(f)) = v^2ff'' + v^2f'^2 + vv'ff' \\ (\mathcal{L}_{\zeta} \mathcal{L}_{\zeta} g)(\partial_x, \partial_x) &= 2vv'' + 4v'^2 \end{aligned}$$

Also, the orthogonal basis of $\mathfrak{X}(M)$ is $\{\partial_t, \partial_x\}$. Thus equation (4.3) becomes

$$\begin{aligned} (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\partial_x, \partial_x) &= 2vv'' + 4v'^2 \\ (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\partial_x, \partial_t) &= 0 \\ (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\partial_t, \partial_x) &= 0 \\ (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} \bar{g})(\partial_t, \partial_t) &= f^2(2u\ddot{u} + 4\dot{u}^2) + 8\dot{u}vf' + 2v^2ff'' + 2v^2f'^2 + 2vv'ff' \end{aligned}$$

Now if $u\partial_t$ and $v\partial_t$ are 2-Killing vector fields on I and \mathbb{R} respectively, then

$$2u\ddot{u} + 4\dot{u}^2 = 2vv'' + 4v'^2 = 0$$

Consequently, ζ is 2-Killing if $f' = 0$. One can get the same result using the definition of 2-Killing vector fields (see Appendix B).

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APPENDIX A. PROOF OF THEOREM 3

Using Proposition 1 and Proposition 3, we get

$$\begin{aligned} (\mathcal{L}_{\zeta} \mathcal{L}_{\zeta} g)(X, Y) &= g(D_{\zeta} D_X \zeta, Y) + g(X, D_{\zeta} D_Y \zeta) - g(D_{[\zeta, X]} \zeta, Y) - g(X, D_{[\zeta, Y]} \zeta) \\ &\quad + 2g(D_X \zeta, D_Y \zeta) \end{aligned}$$

The first term T_1 is given by

$$\begin{aligned}
T_1 &= g(D_\zeta D_X \zeta, Y) \\
&= g\left(D_\zeta \left(D_{X_1}^1 \zeta_1 + \frac{1}{f} \zeta_1(f) X_2 + \frac{1}{f} X_1(f) \zeta_2 + D_{X_2}^2 \zeta_2 - f g_2(X_2, \zeta_2) \nabla f\right), Y\right) \\
&= g\left(D_{\zeta_1}^1 D_{X_1}^1 \zeta_1 + \frac{1}{f} \zeta_1(\zeta_1(f)) X_2 + \frac{1}{f} \zeta_1(X_1(f)) \zeta_2 + \frac{1}{f} \zeta_1(f) D_{X_2}^2 \zeta_2 \right. \\
&\quad \left. - \zeta_1(f) g_2(X_2, \zeta_2) \nabla f - f g_2(X_2, \zeta_2) D_{\zeta_1}^1 \nabla f + \frac{1}{f} (D_{X_1}^1 \zeta_1)(f) \zeta_2 \right. \\
&\quad \left. + \frac{1}{f} \zeta_1(f) D_{\zeta_2}^2 X_2 - \zeta_1(f) g_2(X_2, \zeta_2) \nabla f + \frac{1}{f} X_1(f) D_{\zeta_2}^2 \zeta_2 \right. \\
&\quad \left. - X_1(f) g_2(\zeta_2, \zeta_2) \nabla f + D_{\zeta_2}^2 D_{X_2}^2 \zeta_2 - f g_2(D_{X_2}^2 \zeta_2, \zeta_2) \nabla f \right. \\
&\quad \left. - f g_2(D_{\zeta_2}^2 X_2, \zeta_2) \nabla f - f g_2(X_2, D_{\zeta_2}^2 \zeta_2) \nabla f - g_2(X_2, \zeta_2) (\nabla f)(f) \zeta_2, Y\right)
\end{aligned}$$

and so

$$\begin{aligned}
T_1 &= g_1\left(D_{\zeta_1}^1 D_{X_1}^1 \zeta_1, Y_1\right) + f \zeta_1(\zeta_1(f)) g_2(X_2, Y_2) + f \zeta_1(X_1(f)) g_2(\zeta_2, Y_2) \\
&\quad + f \zeta_1(f) g_2(D_{X_2}^2 \zeta_2, Y_2) - \zeta_1(f) Y_1(f) g_2(X_2, \zeta_2) - f g_2(X_2, \zeta_2) g_1\left(D_{\zeta_1}^1 \nabla f, Y_1\right) \\
&\quad + f (D_{X_1}^1 \zeta_1)(f) g_2(\zeta_2, Y_2) + f \zeta_1(f) g_2\left(D_{\zeta_2}^2 X_2, Y_2\right) - \zeta_1(f) Y_1(f) g_2(X_2, \zeta_2) \\
&\quad + f X_1(f) g_2\left(D_{\zeta_2}^2 \zeta_2, Y_2\right) - X_1(f) Y_1(f) g_2(\zeta_2, \zeta_2) + f^2 g_2\left(D_{\zeta_2}^2 D_{X_2}^2 \zeta_2, Y_2\right) \\
&\quad - f Y_1(f) g_2(D_{X_2}^2 \zeta_2, \zeta_2) - f Y_1(f) g_2\left(D_{\zeta_2}^2 X_2, \zeta_2\right) - f Y_1(f) g_2\left(X_2, D_{\zeta_2}^2 \zeta_2\right) \\
&\quad - f^2 g_2(X_2, \zeta_2) (\nabla f)(f) g_2(\zeta_2, Y_2) \\
&= g_1\left(D_{\zeta_1}^1 D_{X_1}^1 \zeta_1, Y_1\right) + f^2 g_2\left(D_{\zeta_2}^2 D_{X_2}^2 \zeta_2, Y_2\right) \\
&\quad + f \zeta_1(\zeta_1(f)) g_2(X_2, Y_2) + f \zeta_1(X_1(f)) g_2(\zeta_2, Y_2) + f \zeta_1(f) g_2(D_{X_2}^2 \zeta_2, Y_2) \\
&\quad - f \zeta_1(Y_1(f)) g_2(X_2, \zeta_2) + f g_2(X_2, \zeta_2) \left(D_{\zeta_1}^1 Y_1\right)(f) \\
&\quad + f g_2(\zeta_2, Y_2) (D_{X_1}^1 \zeta_1)(f) + f \zeta_1(f) g_2\left(D_{\zeta_2}^2 X_2, Y_2\right) - 2 \zeta_1(f) Y_1(f) g_2(X_2, \zeta_2) \\
&\quad + f X_1(f) g_2\left(D_{\zeta_2}^2 \zeta_2, Y_2\right) - X_1(f) Y_1(f) g_2(\zeta_2, \zeta_2) \\
&\quad - f Y_1(f) g_2(D_{X_2}^2 \zeta_2, \zeta_2) - f Y_1(f) g_2\left(D_{\zeta_2}^2 X_2, \zeta_2\right) - f Y_1(f) g_2\left(X_2, D_{\zeta_2}^2 \zeta_2\right) \\
&\quad - f^2 g_2(X_2, \zeta_2) g_2(\zeta_2, Y_2) (\nabla f)(f)
\end{aligned}$$

Exchanging X and Y we get the second term T_2 and so

$$\begin{aligned}
& T_1 + T_2 \\
&= g(D_\zeta D_X \zeta, Y) + g(D_\zeta D_Y \zeta, X) \\
&= g_1(D_{\zeta_1}^1 D_{X_1}^1 \zeta_1, Y_1) + f^2 g_2(D_{\zeta_2}^2 D_{X_2}^2 \zeta_2, Y_2) + g_1(D_{\zeta_1}^1 D_{Y_1}^1 \zeta_1, X_1) + f^2 g_2(D_{\zeta_2}^2 D_{Y_2}^2 \zeta_2, X_2) \\
&\quad + 2f \zeta_1(\zeta_1(f)) g_2(X_2, Y_2) - 2X_1(f) Y_1(f) g_2(\zeta_2, \zeta_2) - 2f^2 g_2(X_2, \zeta_2) g_2(\zeta_2, Y_2) (\nabla f)(f) \\
&\quad + f \zeta_1(f) g_2(D_{X_2}^2 \zeta_2, Y_2) + f g_2(X_2, \zeta_2) (D_{\zeta_1}^1 Y_1)(f) + f g_2(\zeta_2, X_2) (D_{Y_1}^1 \zeta_1)(f) \\
&\quad + f \zeta_1(f) g_2(D_{Y_2}^2 \zeta_2, X_2) + f g_2(Y_2, \zeta_2) (D_{\zeta_1}^1 X_1)(f) + f g_2(\zeta_2, Y_2) (D_{X_1}^1 \zeta_1)(f) \\
&\quad + f \zeta_1(f) g_2(D_{\zeta_2}^2 X_2, Y_2) - 2\zeta_1(f) Y_1(f) g_2(X_2, \zeta_2) - f Y_1(f) g_2(D_{X_2}^2 \zeta_2, \zeta_2) \\
&\quad + f \zeta_1(f) g_2(D_{\zeta_2}^2 Y_2, X_2) - 2\zeta_1(f) X_1(f) g_2(Y_2, \zeta_2) - f X_1(f) g_2(D_{Y_2}^2 \zeta_2, \zeta_2) \\
&\quad - f Y_1(f) g_2(D_{\zeta_2}^2 X_2, \zeta_2) - f X_1(f) g_2(D_{\zeta_2}^2 Y_2, \zeta_2)
\end{aligned}$$

The third term is given by

$$\begin{aligned}
& T_3 \\
&= g(D_{[\zeta, X]} \zeta, Y) \\
&= g(D_{[\zeta_1, X_1]} \zeta_1 + D_{[\zeta_2, X_2]} \zeta_1 + D_{[\zeta_1, X_1]} \zeta_2 + D_{[\zeta_2, X_2]} \zeta_2, Y) \\
&= g\left(D_{[\zeta_1, X_1]}^1 \zeta_1 + \frac{1}{f} \zeta_1(f) [\zeta_2, X_2] + \frac{1}{f} [\zeta_1, X_1](f) \zeta_2 + D_{[\zeta_2, X_2]}^2 \zeta_2 - f g_2([\zeta_2, X_2], \zeta_2) \nabla f, Y\right) \\
&= g_1(D_{[\zeta_1, X_1]}^1 \zeta_1, Y_1) + f \zeta_1(f) g_2([\zeta_2, X_2], Y_2) + f [\zeta_1, X_1](f) g_2(\zeta_2, Y_2) \\
&\quad + f^2 g_2(D_{[\zeta_2, X_2]}^2 \zeta_2, Y_2) - f g_2([\zeta_2, X_2], \zeta_2) Y_1(f) \\
&= g_1(D_{[\zeta_1, X_1]}^1 \zeta_1, Y_1) + f^2 g_2(D_{[\zeta_2, X_2]}^2 \zeta_2, Y_2) + f \zeta_1(f) g_2([\zeta_2, X_2], Y_2) \\
&\quad + f g_2(\zeta_2, Y_2) [\zeta_1, X_1](f) - f g_2([\zeta_2, X_2], \zeta_2) Y_1(f)
\end{aligned}$$

Exchanging X and Y we get the fourth term T_4 and so

$$\begin{aligned}
& T_3 + T_4 \\
&= g_1(D_{[\zeta_1, X_1]}^1 \zeta_1, Y_1) + g_1(D_{[\zeta_1, Y_1]}^1 \zeta_1, X_1) + f^2 g_2(D_{[\zeta_2, X_2]}^2 \zeta_2, Y_2) + f^2 g_2(D_{[\zeta_2, Y_2]}^2 \zeta_2, X_2) \\
&\quad + f \zeta_1(f) g_2([\zeta_2, X_2], Y_2) + f g_2(\zeta_2, Y_2) [\zeta_1, X_1](f) - f Y_1(f) g_2([\zeta_2, X_2], \zeta_2) \\
&\quad + f \zeta_1(f) g_2([\zeta_2, Y_2], X_2) + f g_2(\zeta_2, X_2) [\zeta_1, Y_1](f) - f X_1(f) g_2([\zeta_2, Y_2], \zeta_2)
\end{aligned}$$

The last term T_5 is given by

$$\begin{aligned}
& 0.5T_5 \\
&= g(D_X \zeta, D_Y \zeta) \\
&= g \left(D_{X_1}^1 \zeta_1 + \frac{1}{f} \zeta_1(f) X_2 + \frac{1}{f} X_1(f) \zeta_2 + D_{X_2}^2 \zeta_2 - f g_2(X_2, \zeta_2) \nabla f, \right. \\
&\quad \left. D_{Y_1}^1 \zeta_1 + \frac{1}{f} \zeta_1(f) Y_2 + \frac{1}{f} Y_1(f) \zeta_2 + D_{Y_2}^2 \zeta_2 - f g_2(Y_2, \zeta_2) \nabla f \right) \\
&= g_1(D_{X_1}^1 \zeta_1, D_{Y_1}^1 \zeta_1) - f g_2(Y_2, \zeta_2) (D_{X_1}^1 \zeta_1)(f) + \zeta_1(f) \zeta_1(f) g_2(X_2, Y_2) \\
&\quad + \zeta_1(f) Y_1(f) g_2(X_2, \zeta_2) + f \zeta_1(f) g_2(X_2, D_{Y_2}^2 \zeta_2) \\
&\quad + \zeta_1(f) X_1(f) g_2(\zeta_2, Y_2) + X_1(f) Y_1(f) g_2(\zeta_2, \zeta_2) + f X_1(f) g_2(\zeta_2, D_{Y_2}^2 \zeta_2) \\
&\quad + f \zeta_1(f) g_2(D_{X_2}^2 \zeta_2, Y_2) + f Y_1(f) g_2(D_{X_2}^2 \zeta_2, \zeta_2) + f^2 g_2(D_{X_2}^2 \zeta_2, D_{Y_2}^2 \zeta_2) \\
&\quad - f g_2(X_2, \zeta_2) (D_{Y_1}^1 \zeta_1)(f) + f^2 g_2(X_2, \zeta_2) g_2(Y_2, \zeta_2) g_1(\nabla f, \nabla f)
\end{aligned}$$

Then

$$\begin{aligned}
(\mathcal{L}_\zeta \mathcal{L}_\zeta g)(X, Y) &= \left(\mathcal{L}_{\zeta_1}^1 \mathcal{L}_{\zeta_1}^1 g_1 \right) (X_1, Y_1) + f^2 \left(\mathcal{L}_{\zeta_2}^2 \mathcal{L}_{\zeta_2}^2 g_2 \right) (X_2, Y_2) \\
&\quad + 4f \zeta_1(f) \left(\mathcal{L}_{\zeta_2}^2 g_2 \right) (X_2, Y_2) + 2f \zeta_1(\zeta_1(f)) g_2(X_2, Y_2) \\
&\quad + 2\zeta_1(f) \zeta_1(f) g_2(X_2, Y_2)
\end{aligned}$$

APPENDIX B. SPACETIME EXAMPLE

In this section we represent the warped product manifold $I_f \times \mathbb{R}$. Using Proposition 1, one can easily get that

- (1) $\nabla_{\partial_x} \partial_x = 0$,
- (2) $\nabla_{\partial_t} \partial_x = \nabla_{\partial_x} \partial_t = \partial_x (\ln f) \partial_t = \frac{f'}{f} \partial_t$ and
- (3) $\nabla_{\partial_t} \partial_t = -f f' \partial_x$

on the warped product manifold $I_f \times \mathbb{R}$. It is clear that

$$[\bar{\zeta}, \partial_t] = -\dot{u} \partial_t \quad [\bar{\zeta}, \partial_x] = -v' \partial_x$$

Also, we have

$$\begin{aligned}
\nabla_{\partial_t} \bar{\zeta} &= -u f f' \partial_x + \frac{1}{f} (\dot{u} f + v f') \partial_t \\
\nabla_{\partial_x} \bar{\zeta} &= v' \partial_x + \frac{1}{f} (u f') \partial_t
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\bar{\zeta}} \nabla_{\partial_t} \bar{\zeta} &= [-u v f f'' - 2u v f'^2 - 2u \dot{u} f f'] \partial_x \\
&\quad + \frac{1}{f} [v^2 f'' + v v' f' + v \dot{u} f' - u^2 f f'^2 + u \dot{u} f] \partial_t \\
\nabla_{\bar{\zeta}} \nabla_{\partial_x} \bar{\zeta} &= (v v'' - u^2 f'^2) \partial_x + \frac{1}{f} (u \dot{u} f' + u v' f' + u v f'') \partial_t
\end{aligned}$$

Finally,

$$\begin{aligned}\nabla_{[\bar{\zeta}, \partial_t]} \bar{\zeta} &= u \dot{u} f f' \partial_x - \frac{1}{f} (\dot{u} v f' + \dot{u}^2 f) \partial_t \\ \nabla_{[\bar{\zeta}, \partial_x]} \bar{\zeta} &= -v'^2 \partial_x - \frac{1}{f} (u v' f') \partial_t\end{aligned}$$

Now we can evaluate the 2-Killing form on $I_f \times \mathbb{R}$ as follows

$$\begin{aligned}(\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} g)(\partial_x, \partial_x) &= 2[vv'' + 2v'^2] \\ (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} g)(\partial_t, \partial_x) &= 0 \\ (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} g)(\partial_x, \partial_t) &= 0 \\ (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{\mathcal{L}}_{\bar{\zeta}} g)(\partial_t, \partial_t) &= 2f^2[u\ddot{u} + 2\dot{u}^2] + 2[v^2 f f'' + v v' f f'] + 8\dot{u} v f f' + 2v^2 f'^2\end{aligned}$$

which is what we have done before.

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